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Foreword

•http://www.springer.com/la/book/9783319569093
•https://www.amazon.com/Understanding-Mathematical-Way-Thinking-Representations/dp/3319569090/ref=sr_1_3?ie=UTF8&qid=1501342420&sr=8-3&keywords=Understanding+the+Mathematical+Way+of+Thinking
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After devoting his time to the study of problems related to learning the concept of infinity in mathematics and to the distinction between demonstration and argumentation, in the early years of the 1990s, Raymond Duval surprised the world of research in the didactics of mathematics with an intense succession of studies into the decisive importance of semiotics in the activity of learning mathematical concepts (Duval 1993).

We had sporadic meetings during that time, particularly during the ICMI Study Conference, on "Perspectives on the Teaching of Geometry for the 21st Century", in Catania in 1995 and, months later, in July 1996 in Seville, at the ICME 8 where I had the opportunity of collaborating with Raymond Duval in the topic group I led (as chief organizer) on "Learning infinity: Infinite processes throughout the curriculum," although by that time his research efforts were focused on semiotics (Duval 1995).

His approach, while drawing from the conceptual bases of Frege, De Saussure, and Peirce, is decidedly revolutionary. The sense he gives to his research is strictly related to the learning of mathematics, and his now world-renowned phrase "There is no noesis without semiosis" has been pronounced by all of us and is one of the phrases most cited in our research world.

Another universal contribution is the idea of the "cognitive learning paradox," in which it is stated that it is paradoxical that the student constructs a mathematical object, by having only semiotic representations of O and not knowledge of O. Moreover, semiotic representations are the only means by which teaching might show O. This is a position with thousands of years of history, since the time of Agustín de Tagaste, who said:

> Cum enim mihi signum datur, si nescientem me invenit cuius rei signum sit, docere me nihil potest: si vero scientem, quid disco per signum?» (For when I am shown a sign, it cannot teach me anything if it finds me ignorant of the reality for which the sign stands; but if it finds me acquainted with the reality, what do I learn from the sign?) (*De magistro*, 10, 115).

In a recent work (D'Amore et al. 2015), we show precisely how this brilliant idea of Raymond results from the itinerary of philosophical-semiotic positions that

began in ancient Greece: I personally communicated this fact during an international conference in Santa Marta, and Raymond expressed great interest in this analysis.

During my visits to Lille, where he lives, Raymond was always generous in every way. So much so that now I can say with certainty that his principal works reflect, in my opinion, only a small part of his real knowledge. He knows how to disseminate this knowledge in many forms during research seminars, in supervising doctoral students, in master's courses, in meetings with teachers, in colloquia with students, and beyond. Those who know him and who hear him speaking at such events can experience how the didactic transposition is valid at every level of schooling and is indeed an exciting subject.

Personally speaking, I think I only definitely understood the differences between Vygotsky and Piaget, thanks to his explanations, one day, in Lille, at his house, while trying, without much success, to prepare a lunch—even after having read all that has been written about this subject. This was also the case as regards my understanding of and the position of De Saussure.

He quotes frequently, and also in this book on pages 27 and 28, the American artist Joseph Kosuth, famous for being one of the first artists of the so-called analytical line (D'Amore 2015a). Among so many works, Kosuth is famous for "One and Three Chairs," made from 1966 onwards in so many versions and currently housed in the best museums in the world. In this, he exposes a real chair of various materials, a photograph of the chair, and a definition of the word "chair" taken from a dictionary.

It is a work of great importance in the world of art history and also interesting to the field of semiotics. I am happy to inform Raymond in these lines that the French artist Bernar Venet, in 1996, also exhibited a work entitled "Tube," which is based on the same idea: the object (a tube) and an axonometric representation (D'Amore 2015b), another indication of this time of operation. I think that for a study of the importance of semiotics in art, it is necessary to start with the Belgian René Magritte (D'Amore 2010).

In this book, Raymond explores his semiotic world as a giant, as only he can do—simply and briefly, but, at the same time, profoundly, learnedly, and concretely.

It proposes the relationship between representation and knowledge, thanks to a revolution within the semiotics (Duval 2006) and based on the role of representation of the knowledge of an object, in general, and of a mathematical object, in particular (Duval 2009a). He also presents the difference, from the cognitive point of view, between sign and representation (Duval 2009b).

Semiotics thus presents itself as a new scheme of knowledge analysis, which requires an in-depth discussion of the three fundamental models of the analysis of signs, each with its contributions and its limits: De Saussure's model (structural analysis of semiotic systems), Peirce's model (classification of the various types of representation), and the model of Frege (the semiotic process that produces new knowledge). (See Raymond Duval's three articles in Duval and Saenz-Ludlow 2016, and my specific comments in the same text).

A central problem has to do with the relationship between mathematical activities and semiotic transformations. In the process of accessing mathematical objects, two epistemological situations are evident, one irreducible to the other. A test (called an opposition) is used with a material object: how to recognize the same object in different representations and how to create correspondence between objects or between representations.

Transformations of semiotic representations are placed by Raymond at the center of mathematical activity. Here, the author is led by convincing examples of great epistemological and didactic force already published in other texts on geometric figures and natural numbers (Duval and Saenz-Ludlow 2016). From these reflections, he concludes that a cognitive analysis of mathematical activity and the functioning of the thought in mathematics are necessary.

We then proceed to the registers of semiotic representation and to the analysis of the cognitive functioning of thought in mathematics: the important difference between codes and registers, the analysis of the types of discursive operations and cognitive functions of natural languages, the relations between thought and language, and the characterization of a semiotic representation register.

This is followed by an analysis of one of the bastions of research in didactics of mathematics: visualization. How do we see a figure? How do we see the transformations of a figure? How does all this work in the didactics of geometry? The examples presented here are portentous and precious.

A complete chapter is devoted to registers. Here, we find considerations about the unity of meaning in mathematics relative to the content of a representation, how mathematical activities vary according to the registers that are put into play, functional variations of the phenomenological modes of production in relation to registers, and how to achieve meaningful and useful analysis of classroom activities.

The power of this work, as with others of the same author, is that the relation between studies, apparently only theoretical, and classroom life is total. He has frequented many classrooms, and I have read his narratives of the activities developed with students. In fact, the theoretical flights that sometimes seem to leave behind the concrete classroom situations are always occasions to recapture living spaces, where the interests are closer to those of the students and the teachers, in order to reach the highest peaks.

These living spaces have no age limits. Some of his considerations are appropriate for children in the first grades of schooling, while others seem to have no reference to age at all. Indeed, I am always surprised to think of the formidable epistemological analysis of his work, which treats the classroom as an experimental environment for the construction of mathematical objects.

His very personal way of seeing, in the specific sense of "seeing," figures is a remarkable contribution to epistemological research in mathematics; I find confirmation in others of his studies (Duval 2016).

The step taken toward the world of art is most impressive, since it provides the scholar with two semiotic fields, the artistic and the mathematical, considered not at all interchangeable in the common sense.

The study of this brief book will undoubtedly be of benefit to those who carry out research in the field of mathematics education, as well as to those who work daily in the world of school and wish to eliminate the gap between the mathematical objects that we intend our students to build cognitively and those that, in reality, the student constructs. This is treated as, as if this was, an epistemological problem, yes, but a problem of a concrete didactical nature.

Raymond himself suggests four different modes which might be adopted while reading this book: a linear reading, the most straightforward and diffuse, from beginning to end; a synoptic transversal reading of recurrent themes; a practical reading, looking for themes and explicit didactic suggestions; or reading it as if it were a cartoon or a book for children, looking only at the figures and following them in their evolution. I personally followed the first modality, and, being fortunate to know in detail these subjects, I appreciated the structure and choice. I then tried to follow the figures, as a child would. I must admit that a certain amount of courage is necessary, but it is undoubtedly an interesting modality of reading.

To the reader, now, is the choice.

Bruno D'Amore

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